COUNTING ZEROS OF CLOSED 1-FORMS

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To the memory of V. A. Rokhlin

Abstract. This paper suggests new topological lower bounds for the number of zeros of closed 1-forms within a given cohomology class. The main new technical tool is the *deformation complex*, which allows to pass to *a singular limit* and reduce the original problem with closed 1-form to a traditional problem with a Morse function. We show by examples that the present approach may provide stronger estimates than the Novikov inequalities. The technique of the paper also applies to study topology of the set of zeros of closed 1-forms under Bott non-degeneracy assumptions

§1. Line bundles, Dirichlet units, and zeros of closed 1-forms

In this section we describe some new topological estimates on the number of zeros of closed 1-forms. They use homology with coefficients in finite dimensional flat vector bundles (compare [N3], [P]), replacing the local systems over the Novikov ring in the well-known Novikov's theory [N1, N2]. We will compute examples (cf. §5) showing that the method of this paper may produce stronger inequalities (and thus predicts existence of a larger number of zeros) than the classical Novikov's approach.

Estimates on the numbers of zeros of closed 1-forms found interesting applications in symplectic topology. They were initiated by J.-C. Sikorav [S1, S2]. We refer also to the work of H. Hofer and D. Salamon [HS].

1.1. Let M be a smooth manifold, and let ω be a closed 1-form on M, $d\omega = 0$. A point $p \in M$ is *a zero* of ω if ω vanishes at this point, i.e. $\omega_p = 0$. A zero $p \in M$ is called *nondegenerate* if ω, viewed as a map $M \to T^*M$, is transversal to the zero section $M \subset T^*M$ of the cotangent bundle. As is well-known, this condition is equivalent to the requirement that in a neighborhood U of p we may write $\omega = df$, where $f : U \to \mathbf{R}$ is a smooth function and p is a non-degenerated (Morse) critical point of f. The *Morse index* of p is well defined (as the Morse index of p as the critical point of f). A closed 1-form ω is called *Morse* if all its zeros are non-degenerate.

1.2. Recall that a complex number $a \in \mathbf{C}^*$ is called *a Dirichlet unit* if it is a unit of the ring of integers of an algebraic number field. Equivalently, a Dirichlet unit is an

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algebraic integer such that its inverse a^{-1} is also an algebraic integer. Dirichlet unit $a \in \mathbf{C}$ is a root of a monic polynomial

$$a^k + \beta_1 a^{k-1} + \beta_2 a^{k-2} + \dots + \beta_k \,=\, 0 \tag{1-1}$$

with $\beta_1, \beta_2, \dots, \beta_{k-1} \in \mathbf{Z}$ and $\beta_k = \pm 1$; this property clearly characterizes the Dirichlet units.

In [F2, F3] I used Dirichlet units to construct an analog of the Lusternik - Schnirelman theory for closed 1-forms (i.e. without assuming that the zeros are all non-degenerate).

Consider a flat complex vector bundle E over M. It is determined by its monodromy, a linear representation of the fundamental group $\pi_1(M, x_0)$ on the fiber E_0 over the base point x_0, given by the parallel transport along loops. For example, a flat line bundle is determined by a homomorphism $H_1(X; \mathbf{Z}) \to \mathbf{C}^*$, where $\mathbf{C}^*$ is considered as a multiplicative abelian group.

Let $\xi \in H^1(X; \mathbf{Z})$ be an integral cohomology class. Given a complex number $a \in \mathbf{C}^*$, we will consider the complex flat line bundle over M with the following property: the monodromy along any loop $\gamma \in \pi_1(M)$ is the multiplication by $a^{\langle \xi, \gamma \rangle}$. We will denote this bundle by a^ξ.

A lattice $\mathcal{L} \subset V$ in a finite dimensional vector space V is a finitely generated subgroup with $\operatorname{rank} \mathcal{L} = \dim_{\mathbf{C}} V$.

We will say that a complex flat bundle $E \to X$ of rank m *admits an integral lattice* if its monodromy representation $\pi_1(X, x_0) \to \mathrm{GL}_{\mathbf{C}}(E_0)$ is conjugate to a homomorphism $\pi_1(X, x_0) \to \mathrm{GL}_{\mathbf{Z}}(\mathcal{L}_0)$, where $\mathcal{L}_0 \subset E_0$ is a lattice in the fiber. This condition is equivalent to the assumption that E is obtained from a local system $\tilde{E}$ of finitely generated free abelian groups over X by tensoring on $\mathbf{C}$.

1.3. Theorem. *Let M be a closed smooth manifold and let $\xi \in H^1(M; \mathbf{Z})$ be an integral cohomology class. Let $E \to M$ be a complex flat bundle admitting an integral lattice. Let $a \in \mathbf{C}^*$ be a complex number, which is not a Dirichlet unit. Then for any closed 1-form ω on M having Morse type zeros and lying in class ξ, the number $c_j(\omega)$ of zeros of ω having index j satisfies*

$$c_j(\omega) \,\geq\, \frac{\dim_{\mathbf{C}} H_j(M; a^\xi \otimes E)}{\dim E}, \qquad j = 0, 1, 2, \dots. \tag{1-2}$$

Moreover,

$$\sum_{i=0}^{j} (-1)^i c_{j-i}(\omega) \,\geq\, \sum_{i=0}^{j} (-1)^i \frac{\dim_{\mathbf{C}} H_{j-i}(M; a^\xi \otimes E)}{\dim E}, \qquad j = 0, 1, 2, \dots. \tag{1-3}$$

A proof of Theorem 1.3 is given in §3.

Remark 1. Theorem 1.3 gives interesting estimates already in the simplest case when E is taken to be the trivial flat line bundle. In this case, however, inequality (1-3) follows from the Novikov inequalities, written in the form

$$\sum_{i=0}^{j} (-1)^i c_{j-i}(\omega) \,\geq\, q_j(\xi) + \sum_{i=0}^{j} (-1)^i b_{j-i}(\xi), \qquad j = 0, 1, 2, \dots, \tag{1-4}$$

cf. [F1], page 48.

We will show by examples (cf. §5) that by applying Theorem 1.3 with different flat bundles E one obtains stronger estimates, than provided by the Novikov inequalities.

Remark 2. It is easy to show that for transcendental $a \in \mathbf{C}$ the Betti number $\dim_{\mathbf{C}} H_i(M; a^\xi)$ equals the Novikov number $b_i(\xi)$ (and in, particular, it is the same for all transcendental a.

Remark 3. Consider the function

$$a \in \mathbf{C}^* \mapsto \dim_{\mathbf{C}} H_j(M; a^\xi \otimes E).$$

Then there exist only finitely many numbers $a_1, a_2, \dots, a_k \in \mathbf{C}^*$ (they are called *jump points*) so that the corresponding Betti number $\dim_{\mathbf{C}} H_j(M; a^\xi \otimes E)$ is the same for any $a \in \mathbf{C}^*$ which is not one of the jump points. Following [BF1], let us denote by $b_j(\xi; E)$ the value of $\dim_{\mathbf{C}} H_j(M; a^\xi \otimes E)$ for a not a jump point. The number $b_j(\xi; E)$ is a generalization of the Novikov number $b_j(\xi)$. For any of the jump points a_s actually holds

$$\dim_{\mathbf{C}} H_j(M; a_s^\xi \otimes E) > b_j(\xi; E),$$

i.e. *the jumps are always positive.*

Suppose that the flat bundle E admits an integral lattice. Then the jump points $a_1, a_2, \dots, a_k$ are *algebraic numbers* (not necessarily algebraic integers). If a jump happens at a number, which is not a Dirichlet unit, then Theorem 1.3 applies and we obtain estimate (3-2), which is stronger, than the inequality

$$c_j(\omega) \geq \frac{b_j(\xi; E)}{\dim E}.$$

Remark 4. *Theorem 1.3 becomes false if we allow $a \in \mathbf{C}^*$ to be a Dirichlet unit.* To explain this, note that any Dirichlet unit $a \in \mathbf{C}$ is an eigenvalue of an integral square matrix $B = (b_{ij})$ with $\det(B) = 1$. We may find a diffeomorphism of a compact smooth manifold $h : F \to F$ so that h induces the matrix B on homology of some dimension k. Consider the mapping torus M, which obtained from $F \times [0, 1]$ by identifying each point $(x, 0)$ with $(h(x), 1)$. The manifold M is a smooth fiber bundle over the circle and so it admits a closed 1-form ω with no critical points, $c_j(\omega) = 0$ for all j. The homology $H_*(M; a^\xi)$ is nontrivial if and only if the number a is an eigenvalue of the monodromy $h_* : H_*(F; \mathbf{C}) \to H_*(F; \mathbf{C})$. Here ξ denotes the cohomology class of ω. Hence, if a is a Dirichlet unit, we may construct M so that $H_*(M; a^\xi) \neq 0$ and class ξ may be realized by a closed 1-form with no critical points.

§2. Morse inequalities for prime ideals

In this section we describe Morse type inequalities for prime ideals in commutative rings, which will be used in the proof of Theorem 1.3. The results of this section are known to experts in commutative algebra, although I am unable to make a reference.

2.1. Let $\mathcal{R}$ be a commutative Noetherian ring. Each prime ideal $\mathfrak{p} \subset \mathcal{R}$ gives a way of assigning Betti numbers to chain complexes over $\mathcal{R}$. Indeed, let C be a chain complex over $\mathcal{R}$

$$0 \to C_m \to C_{m-1} \to \cdots \to C_0 \to 0$$

with finitely generated free $\mathcal{R}$-modules C_i. Given a prime ideal $\mathfrak{p} \in \mathcal{R}$, we will denote by $b_i(C, \mathfrak{p})$ the $\mathfrak{p}$-Betti number of C, i.e.

$$b_i(C, \mathfrak{p}) = \dim_{Q(\mathcal{R}/\mathfrak{p})} H_i(C \otimes_{\mathcal{R}} Q(\mathcal{R}/\mathfrak{p})), \tag{2-1}$$

where $Q(\mathcal{R}/\mathfrak{p})$ denotes the field of fractions of the factor ring $\mathcal{R}/\mathfrak{p}$. Define also *the Poincaré polynomial* corresponding to $\mathfrak{p}$ as

$$\mathcal{P}_{C,\mathfrak{p}}(\lambda) = \sum_{i=0}^{m} \lambda^i b_i(C,\mathfrak{p}). \tag{2-2}$$

Our purpose is to compare the Poincaré polynomials corresponding to two different prime ideals $\mathfrak{p} \subset \mathfrak{q} \subset \mathcal{R}$.

Given two polynomials

$$\mathcal{P}(\lambda) = p_0 + p_1\lambda + \cdots + p_m\lambda^m, \quad \mathcal{Q}(\lambda) = q_0 + q_1\lambda + \cdots + q_{m'}\lambda^{m'}$$

with $p_i, q_i \in \mathbf{Z}$, we will write

$$\mathcal{P}(\lambda) \succeq \mathcal{Q}(\lambda), \tag{2-3}$$

to indicate that the differentce $\mathcal{P}(\lambda) - \mathcal{Q}(\lambda)$ may be represented in the form

$$\mathcal{P}(\lambda) - \mathcal{Q}(\lambda) \;=\; (1+\lambda)\mathcal{T}(\lambda),$$

where $\mathcal{T}(\lambda)$ is a polynomial with non-negative integral coefficients. The relation $\succeq$ defines a partial order on the set of all integral polynomials in λ. It is well known that (2-3) is equivalent to the following sequence of Morse inequalities:

$$\sum_{j=0}^{r}(-1)^j p_{r-j} \geq \sum_{j=0}^{r}(-1)^j q_{r-j}, \quad r = 0, 1, \dots .$$

2.2. Theorem. *Let C be a free finitely generated chain complex over $\mathcal{R}$ and let $\mathfrak{p} \subset \mathfrak{q} \subset \mathcal{R}$ be two prime ideals. Then for the corresponding Poincaré polynomials holds*

$$\mathcal{P}_{C,\mathfrak{q}}(\lambda) \succeq \mathcal{P}_{C,\mathfrak{p}}(\lambda). \tag{2-4}$$

Proof. Applying twice the Euler - Poincaré theorem to the truncated complex

$$0 \to C_r \to C_{r-1} \to \cdots \to C_0 \to 0,$$

tensored by $Q(\mathcal{R}/\mathfrak{p})$ and by $Q(\mathcal{R}/\mathfrak{q})$, we obtain the following identity

$$\begin{gathered}\sum_{j=0}^{r}(-1)^j b_{r-j}(C,\mathfrak{p}) + B_r(\mathfrak{p}) = \\ \sum_{j=0}^{r}(-1)^j \operatorname{rk} C_{r-j} = \\ \sum_{j=0}^{r}(-1)^j b_{r-j}(C,\mathfrak{q}) + B_r(\mathfrak{q})\end{gathered}$$

where $B_r(\mathfrak{p})$ denotes the dimension over the field $Q(\mathcal{R}/\mathfrak{p})$ of the image of the map

$$d : C_{r+1} \otimes Q(\mathcal{R}/\mathfrak{p}) \to C_r \otimes Q(\mathcal{R}/\mathfrak{p}); \tag{2-5}$$

the number $B_r(\mathfrak{q})$ is defined similarly.

Our statement now is equivalent to the inequality $B_r(\mathfrak{q}) \leq B_r(\mathfrak{p})$ for all r.

Suppose that the homomorphism $d : C_{r+1} \to C_r$ is represented by a matrix with entries in $\mathcal{R}$. Then $B_r(\mathfrak{q})$ equals to the maximal size of the minors of this matrix which do not lie in the ideal $\mathfrak{q}$. The number $B_r(\mathfrak{p})$ has the similar descritpion in terms of this matrix. Since $\mathfrak{p} \subset \mathfrak{q}$, we obtain $B_r(\mathfrak{q}) \leq B_r(\mathfrak{p})$. $\square$

§3. Proof of Theorem 1.3

In this section we will describe a proof of Theorem 1.3, which will be based on the Morse inequalities for prime ideals (cf. §2) and on the existence of a *deformation complex*. While proving Theorem 1.3, we will not use the particular construction of the deformation complex C_*; we will only use its properties (i) - (iv), cf. 3.2 below.

3.1. Suppose that we are in conditions of Theorem 1.3. Since the closed 1-form ω has an integral cohomology class, there exists a map $f : M \to S^1$, so that $\omega = f^*(d\theta)$, where $d\theta$ is the standard angular form on the circle. The zeros of ω are precisely the critical points of f. We assume that ω has only nondegenerated zeros, and hence f is a Morse circle valued function on M. Choose a regular value $b \in S^1$ and let N be the result of cutting of M along the codimension one submanifold $V = f^{-1}(b)$. We have a canonical identification map $\Pi : N \to M$. Note that the boundary of N contains two copies of V. We will denote them by $\partial_+ N$ and $\partial_- N$. The notations $\partial_\pm N$ are chosen so that for the normal vector field X on $\partial_+ N$, pointing inside N, holds $\Pi^*\omega(X) > 0$; also, for the normal vector field Y on $\partial_- N$, pointing inside N, holds $\Pi^*\omega(Y) < 0$.

3.2. In §4 we will construct a chain complex C_* (which we call *deformation complex*) with the following properties:

(i) C_* is a free finitely generated chain complex over the polynomial ring $P = \mathbf{Z}[\tau]$, where τ is an indeterminate.

(ii) For any nonzero complex number $a \in \mathbf{C}^*$ the homology $H_i(\mathbf{C}_a \otimes_P C_*)$, as a vector space, is isomorphic to $H_i(M; a^{-\xi} \otimes E)$, where $i = 0, 1, 2, \dots$. Here $\mathbf{C}_a$ denotes $\mathbf{C}$ with the P-module structure, so that τ acts by multiplication on $a \in \mathbf{C}$.

(iii) Let $\mathbf{C}_0$ denote $\mathbf{C}$ with the trivial P-module structure, i.e. τ acts as zero on $\mathbf{C}$. Then the homology $H_i(\mathbf{C}_0 \otimes_P C_*)$ is isomorphic to $H_i(N, \partial_+ N; \Pi^* E)$, where $i = 0, 1, 2, \dots$.

(iv) Let p be a prime number and let $\mathbf{Z}_p$ denote $\mathbf{Z}/p\mathbf{Z}$, considered as a P-module with the trivial (i.e. $\tau = 0$) action of τ. Then the homology $H_i(\mathbf{Z}_p \otimes_P C_*)$, as an abelian group, is isomorphic to $H_i(N, \partial_+ N; \mathbf{Z}_p \otimes \Pi^* \tilde{E})$, where $\tilde{E}$ denotes a local system of free abelian groups on M with $\tilde{E} \otimes \mathbf{C} = E$.

Intuitively, we may view the deformation complex C_* as a polynomial family of complexes $C_*(M; a^{-\xi} \otimes E)$, where $a \in \mathbf{C}^*$ is a parameter, and such that it has a "singular limit" as $a \to 0$, which is described in (iii) and (iv).

The deformation complex C_* depends on the data M, ξ, E and is not unique. We will use only existence of C_*.

3.3. Let $a \in \mathbf{C}^*$ be a nonzero complex number which is not an algebraic integer. Consider the prime ideal $\mathfrak{p}_a \subset P$ consisting of integral polynomials

$$q(\tau) = q_0 + q_1\tau + \dots + q_m\tau^m, \quad q_j \in \mathbf{Z}$$

with $q(a^{-1}) = 0$. If C_* denotes the deformation complex, then, using the property (ii) and the notations, introduced in §2, we obtain

$$\dim_{Q(P/\mathfrak{p}_a)} H_i(Q(P/\mathfrak{p}_a) \otimes_P C_*) = \dim_C H_i(\mathbf{C}_{a^{-1}} \otimes_P C_*) = \dim_{\mathbf{C}} H_i(M; a^{\xi} \otimes E)$$

and therefore

$$\mathcal{P}_{C_*,\mathfrak{p}_a}(\lambda) = \sum_{i=0}^{\dim M} \lambda^i \dim_{\mathbf{C}} H_i(M; a^\xi \otimes E). \tag{3-1}$$

Since we assume that a is not an algebraic integer, the ideal $\mathfrak{p}_a$ contains no polynomials with the free term q_0 equals 1. Hence, we obtain that there exists a prime number p, so that the free terms of all polynomials $q(\lambda)$ lying in $\mathfrak{p}_a$ are divisible by p. In other words,

$$\mathfrak{p}_a \subset \mathfrak{q}, \quad \text{where} \quad \mathfrak{q} = (p) + (\tau) \subset P; \tag{3-2}$$

in other words, the ideal $\mathfrak{q}$ is generated by p and τ. Using property (iv) in 3.2, we obtain

$$\dim_{Q(P/\mathfrak{q})} H_i(Q(P/\mathfrak{q}) \otimes_P C_*) = \dim_{\mathbf{Z}_p} H_i(N, \partial_+ N; \mathbf{Z}_p \otimes \tilde{E}),$$

and thus

$$\mathcal{P}_{C_*,\mathfrak{q}}(\lambda) = \sum_{i=0}^{\dim M} \lambda^i \dim_{\mathbf{Z}_p} H_i(N, \partial_+ N; \mathbf{Z}_p \otimes \tilde{E}). \tag{3-3}$$

By Theorem 2.2 we have

$$\mathcal{P}_{C_*,\mathfrak{q}}(\lambda) \succeq \mathcal{P}_{C_*,\mathfrak{p}_a}(\lambda). \tag{3-4}$$

The form $\Pi^*\omega$ is differential of a Morse function $g : N \to \mathbf{R}$, $dg = \Pi^*\omega$. Function g is constant on each connected component of $\partial_+ N$. The critical points of g lie in the interior of N and they are in 1-1 correspondence with the zeros of ω. Hence, the traditional Morse theory, applied to the cobordism N and Morse function g, gives

$$\sum_{i=0}^{\dim M} c_i(\omega)\lambda^i \succeq (\dim E)^{-1} \cdot \sum_{i=0}^{\dim M} \lambda^i \dim_{\mathbf{Z}_p} H_i(N, \partial_+ N; \mathbf{Z}_p \otimes \tilde{E}). \tag{3-5}$$

Now (3-5), combined with (3-3), (3-4), (3-1), gives

$$\sum_{i=0}^{\dim M} c_i(\omega)\lambda^i \succeq (\dim E)^{-1} \cdot \sum_{i=0}^{\dim M} \lambda^i \dim_{\mathbf{C}} H_i(M; a^\xi \otimes E), \tag{3-6}$$

which is equivalent to (1-3).

We are left to prove Theorem 1.3 assuming that $a^{-1} \in \mathbf{C}^*$ is not an algebraic integer. We will apply the result proven above, where we replace the cohomology class ξ by $-\xi$ and the flat vector bundle E by the flat vector bundle $E^* \otimes \mathfrak{o}_M$, where $\mathfrak{o}_M$ is the orientation bundle of M (i.e. a flat line bunlde such that the monodromy along any loop equals ± 1 depending on whether the orientation of M is preserved or reversed along the loop). Note that E^* as well as $\mathfrak{o}_M$ admit intergral lattices.

This gives

$$\sum_{i=0}^{\dim M} c_i(-\omega)\lambda^i \succeq (\dim E)^{-1} \cdot \sum_{i=0}^{\dim M} \lambda^i \dim_{\mathbf{C}} H_i(M; a^{-\xi} \otimes E^* \otimes \mathfrak{o}_M). \tag{3-7}$$

It is clear that $c_i(-\omega) = c_{n-i}(\omega)$, where $n = \dim M$. Also, from the Poincaré duality we obtain $\dim_{\mathbf{C}} H_i(M; a^{-\xi} \otimes E^* \otimes \mathfrak{o}_M) = \dim_{\mathbf{C}} H_{n-i}(M; a^\xi \otimes E)$. Substitutimg these equalities into (3-7), dividing both sides by λ^n and then replacing the indeterminate λ^{-1} by λ, gives (3-6), which is equivalent to (1-3).

This completes the proof. $\square$

Theorem 1.3 may also be deduced from the main result of [FR].

§4. Construction of the deformation complex

In this section we will construct the deformation complex and prove its properties (i) - (iv), which were used in §3. The construction here is slightly different from [F3], [F4], although the obtained complex is essentially the same.

4.1. The cell decomposition. In this subsection we will describe a cell decomposition of M, related to the given closed 1-form ω. We will repeat some of the constructions of subsection 3.2.

We assume that ω has an integral indivisible cohomology class. Then there exists a smooth map $f : M \to S^1$, so that $\omega = f^*(d\theta)$, where $d\theta$ is the angular form on the circle S^1. Clearly f is a Morse map and the zeros of ω are precisely the critical points of f. Choose a regular value $b \in S^1$, and let N be the result of cutting of M along the codimension one submanifold $V = f^{-1}(b)$. We have a canonical identification map $\Pi : N \to M$. The boundary of N contains two copies of V. We denote them by $\partial_+ N$ and $\partial_- N$. The normal vector field X on $\partial_+ N$, pointing inside N, satisfies $\Pi^*\omega(X) > 0$. Also, for the normal vector field Y on $\partial_- N$, pointing into N, holds $\Pi^*\omega(Y) < 0$.

The form $\Pi^*\omega$ is exact, i.e. it is differential of a Morse function $g : N \to \mathbf{R}$, $dg = \Pi^*\omega$. Each connected component of $\partial_+ N$ consists of points of local minimum of g. The critical points of g, which lie in the interior of N, are in 1-1 correspondence with the zeros of ω and have the same indices. Hence $c_i(g) = c_i(\omega)$ for any i.

Fix a cell decomposition of N so that $V = \partial_+ N$ and $\partial_- N$ be subcomplexes and so that the natural homeomorphism $J : \partial_+ N \to \partial_- N$ be a cellular isomorphism.

The manifold M is homeomorphic to the factor-space $N/\sim$, where we identify each pair of points $v \in \partial_+ N$ and $J(v) \in \partial_- N$. In order to give the factor-space $N/\sim$ a CW-structure, consider a cylinder $V \times [0,1]$ having the standard cell-decomposition: for each i-dimensional cell e of V we have two i-dimensional cells $e \times 0$ and $e \times 1$ and one $(i+1)$-dimensional cell $e \times I$ of the cylinder $V \times I$. Glue each point $(v,1)$ of the top face of the cylinder with the point $v \in V \subset N$; also, glue each point $(v,0)$ of bottom face of the cylinder with the point $J(v) \in N$. As the result we obtain a CW-decomposition

$$M = (V \times I \cup N)/\sim . \tag{4-1}$$

4.2. The chain complex. Here we will calculate the cellular chain complex $C_*(\tilde{M})$ of the universal covering $\tilde{M}$ of the CW-complex (4-1). It is a complex of free left $\mathbf{Z}\pi$-modules, where π is the fundamental group of M.

In general the complement $M - V$ may be disconnected. Let us choose base points $v_1, v_2, \dots, v_l$, one for each connected component of $M - V$. For each $j = 1, 2, \dots, l$ we may find a smooth path $\mu_j : [0,1] \to M$, such that $\mu_j(0) = v_1$, $\mu_j(1) = v_j$ and μ_j has intersection number zero with V, i.e. $\mu_j \cdot V = 0$. Note that V has a fixed orientation of its normal bundle. To construct μ_j, first connect the points v_1 and v_j by an arbitrary path $\overline{\mu}_j$ in M, and then set $\mu_j = \overline{\mu}_j - (\overline{\mu}_j \cdot V)\delta$, where δ is a closed loop in M such that $\delta \cdot V = 1$. Such loop δ exists since we assume that the cohomology class $\xi = [\omega] \in H^1(M; \mathbf{Z})$ is indivisible.

Each cell of X can be lifted into the covering $\tilde{M}$, and all possible lifts are parameterized by the elements of the fundamental group $\pi = \pi_1(M, v_1)$. In order to fix the lifts of the cells of M, we will describe for each cell $e \subset M$ a path γ_e in M, starting

from the base point v_1 and leading to an internal point of e. We will call γ_e *the tail of* e. After an arbitrary choice of a lift of the base point v_1, we will obtain lifts of all the cells in the covering $\tilde{M}$, determined (in an obvious way) by the choice of the tails.

For the cells $e \subset N$ we will choose their tails as follows. Assume that e lies in the component of N containing v_j. Then we set $\gamma_e = \mu_j \sigma_e$, where σ_e is a path in N connecting v_j with an internal point of e.

The tail of cells of the form $e \times I \subset V$, where $e \subset V$, are constructed as follows. First travel along the existing tail γ_e of cell e, which leads from the base point v_1 to an internal point of $e \subset V = V \times 1 \subset N$, and then drop slightly down to an internal point of the cell $e \times I$.

After the choice of tails as above we obtain a free basis of $C_*(\tilde{M})$ (over the group ring $\mathbf{Z}\pi$) formed by the lifts of the cells of M.

The boundary homomorphism of $C_*(\tilde{M})$, applied to a cylindrical generator $e \times I$ is given by the formula

$$d(e \times I) = d(e) \times I + (-1)^i [e \times 1 - e \times 0], \quad i = \dim e. \tag{4-2}$$

Here $e \times 1$ can be identified with e. The generator $e \times 0 \in C_*(\tilde{M})$ is supported by the cell $J(e) \subset \partial_- N \subset N$, however it has a different tail. Indeed, it is clear that the tail of the cell $e \times 0$ has intersection number -1 with V, because it first travels along the tail of $e \times I$, and then arrives (staying inside $e \times I$) at the face $e \times 0 \simeq J(e)$ of $e \times I$. Hence we may rewrite (4-2) as follows

$$d(e \times I) = d(e) \times I + (-1)^i [e - g \cdot J(e)], \tag{4-3}$$

where

$$g \in \pi = \pi_1(M, v_1), \quad \xi(g) = -1. \tag{4-4}$$

Recall that $\xi \in H^1(M; \mathbf{Z})$ denotes the cohomology class of ω.

4.3. The deformation complex. Let $\mathbf{Z}\pi_- \subset \mathbf{Z}\pi$ denote the subring of the group ring generated by the group elements $g \in \pi$ with $\xi(g) \leq 0$. From the description of the chain complex $C_*(\tilde{M})$ (and in particular, from (4-3), (4-4)) we see that the cells of M with their tails specified above, generate a free chain complex C' over $\mathbf{Z}\pi_-$ such that

$$\mathbf{Z}\pi \otimes_{\mathbf{Z}\pi_-} C' \;=\; C_*(\tilde{M}).$$

Let $\tilde{E} \to M$ be a local system of free abelian groups so that $\tilde{E} \otimes \mathbf{C} = E$; it exists because we assume that flat bundle E admits an integral lattice. We have the monodromy representation

$$\mathrm{Mon}_{\tilde{\mathrm{E}}} : \pi \to \mathrm{GL}(m; \mathbf{Z}), \quad m = \mathrm{rank}(\tilde{E}). \tag{4-5}$$

Here for $g, g' \in \pi$ holds $\mathrm{Mon}_{\tilde{\mathrm{E}}}(gg') = \mathrm{Mon}_{\tilde{\mathrm{E}}}(g')\mathrm{Mon}_{\tilde{\mathrm{E}}}(g)$, i.e. (4-5) is an anti-isomorphism. We obtain a ring anti-homomorphism $\mathbf{Z}\pi_- \to \mathrm{Mat}(m; P)$, $\quad P = \mathbf{Z}[\tau]$, where for $g \in \pi$ with $\xi(g) \leq 0$ we set $g \mapsto \tau^{-\xi(g)} \mathrm{Mon}_{\tilde{\mathrm{E}}}(g)$. This defines a structure of right $\mathbf{Z}\pi_-$-module on P^m, commuting with the obvious left P-action. Hence, P^m becomes a $(P, \mathbf{Z}\pi_-)$-bimodule. We may fanally define the deformation complex C_* as

$$C_* = P^m \otimes_{\mathbf{Z}\pi_-} C'. \tag{4-6}$$

4.4. Now we check that the deformation complex (4-6) satisfies the conditions (i) - (iv) of subsection 3.2. Indeed, (i) is obvious. To show (ii), note that for $a \in \mathbf{C}^*$ we have an isomorphism

$$\mathbf{C}_a \otimes_P C_* = \mathbf{C}_a \otimes_P (P^m \otimes_{\mathbf{Z}\pi_-} C') \simeq (\mathbf{C}_a \otimes_{\mathbf{Z}} \mathbf{Z}^m) \otimes_{\mathbf{Z}\pi} C_*(\tilde{M}) \tag{4-7}$$

and $\mathbf{C}_a \otimes_{\mathbf{Z}} \mathbf{Z}^m \simeq \mathbf{C}^m$ is a right $\mathbf{Z}\pi$-module with respect to the action

$$g \mapsto a^{-\xi(g)} \operatorname{Mon}_E(g) \in \mathrm{GL}(m; \mathbf{C}).$$

This clearly implies (ii).

Let us now prove (iv). We have

$$\mathbf{Z}_p \otimes_P C_* \simeq \mathbf{Z}_p \otimes_P (P^m \otimes_{\mathbf{Z}\pi_-} C') \simeq \mathbf{Z}_p^m \otimes_{\mathbf{Z}\pi_-} C', \tag{4-8}$$

where the right action of $\mathbf{Z}\pi_-$ on $\mathbf{Z}_p^m$ is given by

$$g \mapsto \begin{cases} \operatorname{Mon}_{\tilde{\mathrm{E}}}(g), & \text{if} \quad \xi(g) = 0 \\ 0 \quad \text{if} \quad \xi(g) < 0, \end{cases} \tag{4-9}$$

where $g \in \pi$. From (4-3), (4-4) we see that complex (4-8) will have as its basis the cells of N and cells of the form $e \times I$, one for each cell e of V. The boundary map of (4-8) acts on the cylindrical generators $e \times I$ as follows

$$d(e \times I) = \partial e \times I + (-1)^{\dim e} \cdot e \tag{4-10}$$

(the last term in (4-3) disappeares). This implies that all the cells $e \subset V$ and $e \times I \subset V \times I$ generate a subcomplex $\mathfrak{N}_p$ of the chain complex (4-8). Moreover, the map $e \mapsto (-1)^{\dim e} e \times I$ is a chain contraction of $\mathfrak{N}_p$ (because of (4-10)).

Hence we obtain an isomorphism $H_i(\mathbf{Z}_p \otimes_P C_*) \simeq H_i((\mathbf{Z}_p \otimes_P C_*)/\mathfrak{N}_p)$ and the homology of the factor-complex $(\mathbf{Z}_p \otimes_P C_*)/\mathfrak{N}_p$ is clearly $H_i(N, V; \mathbf{Z}_p \otimes \Pi^* \tilde{E})$. This proves (iv).

The proof of (iii) is similar. We will not use (iii) in this paper.

§5. An example

In this section we show that inequalities (1-2), (1-3) produce stronger estimates than the Novikov inequalities. The example, which is described here, is a modification of example 1.7 in [BF].

5.1. Let $k_N \subset S^3$ be the connected sum of N copies of the trefoil knot. Let X be closed 3-dimensional manifold obtained by 0-surgery on k_N. Then $H^1(X; \mathbf{Z}) \simeq \mathbf{Z}$; we will denote by $\eta \in H^1(X; \mathbf{Z})$ a generator.

Recall that the Alexander polynomial of the trefoil is $\Delta(\tau) = \tau^2 - \tau + 1$. Its two roots we denote by $b, b^{-1} \in \mathbf{C}$; they are Dirichlet units.

Consider the rank 2 flat vector bundle $F \to X$, where $F \simeq b^\eta \oplus b^{-\eta}$. From the knot theory [R] we know that

$$\dim_{\mathbf{C}} H_1(X; F) = 2N, \quad \dim_{\mathbf{C}} H_1(X; \mathbf{C}) = 0. \tag{5-1}$$

Let us show that F admits an integral lattice. The monodromy representation of F is a 2-dimensional vector space with a basis e_1, e_2, on which the meridian $\gamma \in \pi_1(X)$ of k_N acts as follows

$$e_1 \mapsto b e_1, \quad e_2 \mapsto b^{-1} e_2.$$

Hence the vectors $e_1 + e_2$ and $b e_1 + b^{-1} e_2$ form a basis, and the action of γ in this basis is represented by an integral matrix $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

5.2. Consider the 3-dimensional manifold given as the connected sum

$$M = X\#(S^1 \times S^2).$$

Thus $M = M_+ \cup M_-$, where $M_+ \cap M_- = S^2$ and

$$M_+ = X - \{\text{disk}\}, \quad M_- = (S^1 \times S^2) - \{\text{disk}\}.$$

Consider a flat vector bundle $E \to M$ such that $E|_{M_+} \simeq F|_{M_+}$ and $E|_{M_-}$ is trivial. Let $\xi \in H^1(M; \mathbf{Z})$ be a class such that $\xi|_{M_+} = 0$ and $\xi|_{M_-}$ is a generator.

We want to compare $\dim_{\mathbf{C}} H_1(M; a^\xi)$ with $\dim_{\mathbf{C}} H_1(M; a^\xi \otimes E)$ for different $a \in \mathbf{C}^*$. Using the Mayer-Vietoris sequence, we obtain

$$\dim_{\mathbf{C}} H_1(M; a^\xi) = \dim_{\mathbf{C}} H_1(M_+; \mathbf{C}) + \dim_{\mathbf{C}} H_1(M_-; a^\xi),$$

and hence using (5-1) we obtain that $\dim_{\mathbf{C}} H_1(M; a^\xi) = 0$ for $a \neq 1$. All Novikov numbers $b_i(\xi) = 0 = q_i(\xi)$ vanish. However, the same argument as above, using (5-1) and the Mayer-Vietoris sequence, yields

$$\dim_{\mathbf{C}} H_1(M; a^\xi \otimes E) = 2N.$$

This example shows that an appropriate choice of flat bundle E, appearing in inequalities (1-2) and (1-3), may provide stronger estimates, than when taking E to be the trivial bundle.

§6. Closed 1-forms with non-isolated zeros

In this section we study the topology of the set of zeros of closed 1-forms under Bott type [B] nondegeneracy condition. First inequalities of this type were obtained in [BF].

6.1. Let ω be a smooth closed real valued 1-form on M, $d\omega = 0$, which is assumed to have zeros *non-degenerate in the sense of R.Bott* [B]. This means that the set of points of $C \subset M$, where the form ω vanishes, form a submanifold of M, and the *Hessian* of ω is *non-degenerate on the normal bundle to* C.

If N is a small tubular neighborhood of C in M, then the integral $\int_\gamma \omega$ vanishes along any loop $\gamma \subset N$ (since γ is homologous to a curve in C). Thus there exists a unique real valued smooth function f on N such that $df = \omega_{|N}$ and $f_{|C} = 0$. The *Hessian* of ω is then defined as the Hessian of f.

Let $\nu(C)$ denote the normal bundle of C in M. Note that $\nu(C)$ may have different dimension over different connected components of C. Since Hessian of ω is non-degenerate, the bundle $\nu(C)$ splits into the Whitney sum of two subbundles

$$\nu(C) \;=\; \nu^+(C) \;\oplus \nu^-(C),$$

such that the Hessian is strictly positive on $\nu^+(C)$ and strictly negative on $\nu^-(C)$. Here, the dimension of the bundles $\nu^+(C)$ and $\nu^-(C)$ over different connected components of C may be different.

For every connected component Z of set C, the dimension of the bundle $\nu^-(C)$ over Z is called the *index* of Z and is denoted by $\operatorname{ind}(Z)$. Let $o(Z)$ denote the *orientation bundle of* $\nu^-(C)_{|Z}$, considered as a local system with fiber $\mathbf{Z}$.

The following is a generalization of Theorem 1.3 for closed 1-forms with non-isolated zeros.

6.2. Theorem. *Let ω be a closed 1-form on M having Bott type zeros and representing an integral cohomology class $\xi = [\omega] \in H^1(M;\mathbf{Z})$. Let $\tilde{E} \to M$ be a local system of free abelian groups. Let $a \in \mathbf{C}^*$ be a complex number, not an algebraic integer; let p be a prime number with the property that any integral polynomial $q(\tau)$ with $q(a) = 0$ has top coefficient divisible by p. Then*

$$\begin{aligned} &\sum_Z \sum_{i=0}^{n} \lambda^{\mathrm{ind}(Z)+i} \dim_{\mathbf{Z}_p} H_i(Z; \mathbf{Z}_p \otimes \tilde{E}_{|Z} \otimes o(Z)) \succeq \\ &\succeq \sum_{i=0}^{n} \lambda^i \dim_{\mathbf{C}} H_i(M; a^\xi \otimes \tilde{E}). \end{aligned} \tag{6-1}$$

In the first sum Z runs over all connected components of the set of zeros of ω.

Note that for transcendental a the prime number p may be taken arbitrarily. In the case of transcendental $a \in \mathbf{C}^*$ inequalities (6-1) turn into the inequalities of [BF], i.e. with $\mathbf{C}$ replacing $\mathbf{Z}_p$ in the first sum (6-1). In fact, the method of the proof, suggested in the present paper, also proves the main result of [BF] (by using property (iii) in subsection 3.2 of the deformation complex). In [BF] the proof used the method of Witten deformation and analytical tools.

Proof. We simply repeat all the arguments of the proof of Theorem 1.3 given in §3, using the Morse inequalities for prime ideals and the deformation complex. On the last stage instead of inequality (3-5) we use now the following inequality

$$\begin{aligned} &\sum_Z \sum_{i=0}^{n} \lambda^{\mathrm{ind}(Z)+i} \dim_{\mathbf{Z}_p} H_i(Z; \mathbf{Z}_p \otimes \tilde{E}_{|Z} \otimes o(Z)) \succeq \\ &\succeq \sum_{i=0}^{n} \lambda^i \dim_{\mathbf{Z}_p} H_i(N, \partial_+ N; \mathbf{Z}_p \otimes \tilde{E}), \end{aligned} \tag{6-2}$$

which is just a slight generalization of the well-known inequality of Bott [B] with $\mathbf{Z}_p$ coefficients, applied to the manifold with boundary N and to Bott function $g : N \to \mathbf{R}$, cf. §3. $\square$

§7. Classes of higher rank

In this section we announce a generalization of Theorem 1.3 for cohomology classes ξ of higher rank.

7.1. Let M be a manifold. We will denote by H the first homology group $H_1(M;\mathbf{Z})$. Let $\xi \in H^1(M, \mathbf{R})$ be a real cohomology class. It can be viewed as a homomorphism $\xi : H_1(M;\mathbf{Z}) = H \to \mathbf{R}$; we denote by $\ker(\xi)$ the kernel. Given a polynomial $p \in \mathbf{Z}[H]$, one defines two numbers $d_\xi(p)$ (*the ξ-degree of p*) and $v_\xi(p)$ (*the ξ-top coefficient*) as follows. Let $p = \sum_{j=1}^n \beta_j h_j$, where $\beta_j \in \mathbf{Z}$ and $h_j \in H$. Then $d_\xi(p)$ is defined as the maximal number $d = d_\xi(p) \in \mathbf{R}$ such that the sum $v_\xi(p) = \sum \beta_j$, taken over all j with $\langle \xi, h_j \rangle = d$, is nonzero.

Let $L \to M$ be a complex flat line bundle. We will assume that the monodromy of L is trivial along any loop in M representing a homology class in $\ker(\xi)$. L determines a *monodromy homomorphism*

$$\mathrm{Mon_L} : \mathbf{Z}[H] \to \mathbf{C}, \tag{7-1}$$

assigning to any $h \in H$ the monodromy of L along h. We will denote by $\mathcal{I}_L \subset \mathbf{Z}[H]$ the kernel of the homomorphism $\mathrm{Mon_L}$.

7.2. Definition. *(A) We will say that a flat complex line bundle $L \to M$ is a ξ-algebraic integer if (i) the monodromy of L is trivial along any loop in M representing a homology class in* $\ker(\xi)$*; and (ii) the ideal $\mathcal{I}_L$ contains a polynomial $p \in \mathcal{I}_L$ with* $v_\xi(p) = \pm 1$.

(B) We will say that a complex flat line bundle $L \to M$ is a ξ-Dirichlet unit if L and the dual flat line bundle L^* are ξ-algebraic integers.

The following statement generalizes Theorem 1.3 on forms with arbitrary cohomology classes.

7.3. Theorem. *Let M be a closed smooth manifold and let $\xi \in H^1(M; \mathbf{R})$ be a real cohomology class. Let $E \to M$ be a flat complex vector bundle admitting an integral lattice. Let $L \to M$ be a flat complex line bundle, which is not a ξ-Dirichlet unit. Then for any closed 1-form ω on M having Morse zeros and lying in the class ξ, the number $c_j(\omega)$ of zeros of ω having index j satisfies*

$$c_j(\omega) \ge \frac{\dim_{\mathbf{C}} H_j(M; L \otimes E)}{\dim E}, \qquad j = 0, 1, 2, \dots. \tag{7-2}$$

The proof will be published elsewhere. It uses the Morse inequalities for prime ideals and an analog of the deformation complex for forms of higher rank > 1.

References

[BF] M. Braverman and M. Farber, *Novikov type inequalities for differential forms with non-isolated zeros*, Math. Proc. Camb. Phil. Soc. **122** (1997), 357 - 375.

[B] R. Bott, *Non degenerate crititcal manifolds*, Annals of Math **60** (1954), 248 - 261.

[F1] M. Farber, *Exactness of the Novikov inequalities*, Functional Anal. Appl. **19** (1985), 40–48.

[F2] M. Farber, *Dirichlet units and critical points of closed 1-forms.*, To appear in "C. R. Acad. Sci. Paris".

[F3] M. Farber, *Lusternik - Schnirelman theory for closed 1-forms*, Preprint math.DG/9811113 (1998).

[F4] M. Farber, *Topology of closed 1-forms and their critical points*, Preprint math.DG/9811173.

[FR] M. Farber and A. Ranicki, *The Morse - Novikov theory of circle-valued functions and non-commutative localization*, Preprint math.DG/9811173.

[HS] H. Hofer and D. Salamon, *Floer homology and Novikov rings*, Floer memorial volume, Birkhauser, 1995, 483 - 524.

[M] J. Milnor, *Morse theory*, Annals of Mathematics Studies, vol. 51, Princeton Univ. Press, 1963.

[N1] S.P. Novikov, *Multi-valued functions and functionals. An analogue of Morse theory.*, Soviet Math. Doklady **24** (1981), 222–226.

[N2] S.P. Novikov, *The Hamiltonian formalism and a multi-valued analogue of Morse theory*, Russian Math. Surveys **37:5** (1982), 1–56.

[N3] S.P. Novikov, *Bloch homology, critical points of functions and closed 1-forms*, Soviet Math. Dokl. **33** (1986), 551–555.

[P] A. Pazhitnov, *Morse theory of closed 1-forms*, Lecture Notes in Math **1474** (1991).
[R] D. Rolfsen, *Knots and links*, 1976.
[Sh] V.V. Sharko, *Functions on Manifolds*, Translations of Mathematical Monographs, vol. 131, American Mathematical Society, 1993.
[S1] J.-C. Sikorav, *Thèse*.
[S2] J.-C. Sikorav, *Un problem de disjonction par isotopic symplectique dans un fibré cotangent*, Ann. Scient. Ecole Norm. Sup. **19** (1986), 543-552.

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